Duality and Symmetries in the Zoo of Fourier Transforms

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For my students in ELEC242 at Rice University.

I promise not to test you on knowledge of this particular document, as I am not considering it to be official course material. Hopefully you enjoy it anyway. –*tmr*

Abstract

When learning Fourier analysis in a first course on signals and systems, it can be overwhelming to keep track of the Four Fourier transforms: the CT Fourier series, the DT Fourier series, the CT Fourier transform, and the DT Fourier transform. Additionally, these are often not related in a very concrete way when notions of sampling and approximation are studied. The goal of this document is to consolidate all of these things, and show how each transform can be built out of any of the other transforms. Conceptually, then, if you have a handle on one of them, you have a handle on all of them.

The approach of this document will not focus on numerical calculations. We will completely ignore sign changes, time reversals, conjugations, convergence conditions, etc. Working out the numerical details is part of learning how to really use Fourier analysis in practice; all we are doing here is providing scaffolding to facilitate that understanding.

1 The Setup

The first part of the course was devoted to getting a handle on the basics of linear time-invariant systems. One day in class, we made the observation that complex exponentials are *eigenfunctions* of said systems, that is to say, when you input a complex exponential into an LTI system, the output is just a scaled version of that complex exponential. So, if we can break a signal down as a sum (or integral, perhaps) of complex exponentials, we should be able to very easily understand how *any* LTI system will behave with respect to that signal. This led us naturally to the Fourier transform, which considers the special case of complex exponentials with purely imaginary exponents. We spent some time building up a "zoo" of these Fourier transforms, each one designed for a different type of signal.

1.1 Four Fourier Transforms

The first one was the *continuous-time (CT) Fourier series* (CTFS), which represents periodic signals in continuous time as a sequence of coefficients. As notation, we might call a CT signal with period T by the name $x_T(t)$, and write it something like this:

$$x_T(t) \stackrel{FS}{\longleftrightarrow} a[k].$$

We have used the notation $a[k]$ here for the Fourier series rather than the usual a_k, but we will find this useful for unifying all of the representations.

Following this, we moved to the *discrete-time (DT) Fourier series* (DTFS), which represents periodic signals in discrete time, once again, as a sequence of coefficients. This time, however, the sequence of coefficients is periodic, much like the original signal. Let us denote a DT signal with period N by $x_N[n]$, so that we can write the DT Fourier series like this:

$$x_N[n] \stackrel{FS}{\longleftrightarrow} a_N[k].$$

Notice how we have denoted the Fourier series in discrete-time by $a_N[k]$; this reflects the fact that the DT Fourier series is periodic with the same period as the input signal.

After defining these two transformations for periodic signals, we then asked how we could represent non-periodic signals using similar techniques. This led us to our most general transformation: the Fourier transform. Let us continue to ignore questions about convergence, existence, and so on, and plow on ahead.

The *CT Fourier transform* (CTFT) has the most remarkable symmetry of all of the transforms we have considered: it turns signals in continuous-time to functions[1] of a real (continuous!) variable. For a CT signal $x(t)$, we write this as

$$x(t) \stackrel{FT}{\longleftrightarrow} X(j\omega),$$

where ω denotes real-valued frequency. Unlike the previous two transforms, the "type" of the representation doesn't seem to change too much: both $x(t)$ and its transform $X(j\omega)$ are functions of a real variable.

This, of course, is not the case for the *DT Fourier transform* (DTFT), which maps general DT signals to periodic functions of a real variable. For a DT signal $x[n]$, we write this as

$$x[n] \stackrel{FT}{\longleftrightarrow} X(e^{j\omega}).$$

Our choice of notation for the transform X is perhaps the most interesting so far: we think of it as a function of $e^{j\omega}$ for a real variable ω. Clearly, $e^{j\omega}$ is a periodic function of the real variable ω with period 2π, so the transform is a periodic function of a real variable with period 2π as well.

[1] I won't call them "signals," as we like to reserve that term for functions in the time-domain.

1.2 Summary

It will be useful to write down all Four of the Fourier transforms together, which will hopefully prompt us to notice some similarities between them:

$$\begin{aligned} x_T(t) &\xleftrightarrow{FS} a[k] \\ x_N[n] &\xleftrightarrow{FS} a_N[k] \\ x(t) &\xleftrightarrow{FT} X(j\omega) \\ x[n] &\xleftrightarrow{FT} X(e^{j\omega}). \end{aligned}$$

If you would like to stop here, writing these down is probably a good idea in preparation for your next exam. However, the rest of this document will explore the relationships *between* these four rows, so I suggest reading onward if you are interested.

2 The Hook

Each of the Fourier transforms has a particular formula for analysis and synthesis, which we have discussed in great detail in class. We will cast those aside for now, and look simply at the type of function on either side of the transform above. This will lead us to notice some pleasant symmetries and relationships between these transforms, illuminating the notion that all of these are, in the end, doing the same thing.

2.1 Continuous Circles

At the end of the discussion of the DTFT, we pointed out how the the Fourier transform of a DT signal $x[n]$ is denoted by $X(e^{j\omega})$, where ω is a real variable. Notably, since $e^{j\omega}$ is a periodic function of ω with period 2π, so is $X(e^{j\omega})$.

Let us dig deeper into this. Remember Euler's formula: $e^{j\omega} = \cos(\omega) + j\sin(\omega)$. This tells us that as we vary ω, the function $e^{j\omega}$ plots out a circle in the complex plane.

Exercise 1: The unit circle

Plot out the function $e^{j\omega}$ on the complex plane. See that it is the same thing as the unit circle.

So, noting that $e^{j\omega}$ is a codeword for "the unit circle," we can think of $X(e^{j\omega})$ as "a function on the unit circle." Perhaps we are only used to functions being defined on the real line, or maybe even on the integers (think, CT signals and DT signals, for example). But functions can be defined on anything! In this case X takes in a point on the circle as input, and produces some complex number as output. As useful notation, let us use the symbol S^1 to denote the circle, and then use the notation $X : S^1 \to \mathbb{C}$ to tell ourselves what sort of function the DTFT of a DT signal is, namely, a function that maps the *unit circle* to the *complex numbers*.

We can do the exact same thing for periodic signals in continuous time. A periodic signal $x_T(t)$ is such that for all t, we have $x_T(t+T) = x_T(t)$, where T denotes the period. In our treatment of periodic signals, we convince ourselves that we only really care about the signal as it is defined over a single period: this is a good idea! In fact, it allows us to pretend that a periodic signal is a function on the circle, just like the DTFT. Namely, think of x_T in the following way. For a given point $(\cos(\theta), \sin(\theta))$ on the circle, define $\tilde{x}_T(\theta)$ as $x_T(\theta T/2\pi)$.

Exercise 2: Continuous circular function

Convince yourself that $\tilde{x}_T$ and x_T are "the same," in some sense.

The "continuous-time circular signal" $\tilde{x}_T(\theta)$ essentially takes $x_T(t)$ and coils it around the circle. Although the real line is infinite in length and the circle has finite circumference, we can do this without any issues, since x_T is periodic. In fact, notice that $x_T(t) = \tilde{x}_T(t2\pi/T)$ for any t. Using this, we can once again think of a periodic signal as a function $\tilde{x}_T : S^1 \to \mathbb{C}$.

2.2 Discrete Circles

Thinking of periodic functions as being defined on the circle is not restricted to continuous time. Take a look at the DT Fourier series:

$$x_N[n] \overset{FS}{\longleftrightarrow} a_N[k].$$

The DT Fourier series turns periodic DT signals into periodic functions on the integers. We often think of DT signals as approximations of CT signals, so it is only natural to try and fit this into the idea of a function on the circle like before. But the circle is continuous, not discrete, so this poses a bit of a problem.

The solution lies in discretizing the circle. The circle S^1 is given by the set of points $(\cos(\theta), \sin(\theta))$. Equivalently, it is the set of all complex point $e^{j\omega}$ for real-valued ω. To build a "discrete circle," we just discretize the θ or the ω parameter. For the period N, define the N-discrete circle as

$$S_N^1 = \{e^{j2\pi k/N}, k = 0, \pm 1, \pm 2, \ldots\}.$$

Exercise 3: Discrete unit circle

Plot out S_N^1 on the complex plane, and see that it is indeed a discretization of the circle S^1. Count how many points S_N^1 consists of.

Although we have defined S_N^1 as a seemingly infinite set of complex numbers (as we consider infinitely many values of k), the DT periodicity of $e^{j2\pi k/N}$ implies that S_N^1 consists of only N points, evenly spaced around the unit circle. Great! Now we can think of both sides of the DTFS as functions on a discretized circle. Namely, build two "discrete-time circular functions" in the following way:

$$\tilde{x}_N[2\pi n/N] = x_N[n]$$
$$\tilde{a}_N[2\pi k/N] = a_N[k].$$

Exercise 4: Discrete circular function

Convince yourself that $\tilde{x}_N$ and x_N are "the same," in some sense. Do the same for $\tilde{a}_N$ and a_N.

With this idea in hand, we can now treat periodic DT signals such as x_N as functions $\tilde{x}_N : S^1_N \to \mathbb{C}$, and similarly treat their DT Fourier series $a_N[k]$ as functions $\tilde{a}_N : S^1_N \to \mathbb{C}$.

3 The Sting

In The Hook, we showed how periodic functions can really be thought of as functions on the circle, whether discrete or continuous. Let us rewrite our chart of transforms again, with some different notation.

$$\begin{aligned}
(\mathbb{R} \to \mathbb{C}) &\xleftrightarrow{FT} (\mathbb{R} \to \mathbb{C}) \\
(S^1 \to \mathbb{C}) &\xleftrightarrow{FS} (\mathbb{Z} \to \mathbb{C}) \\
(\mathbb{Z} \to \mathbb{C}) &\xleftrightarrow{FT} (S^1 \to \mathbb{C}) \\
(S^1_N \to \mathbb{C}) &\xleftrightarrow{FS} (S^1_N \to \mathbb{C}).
\end{aligned}$$

Woah! What happened here? Where did x go? What we have done with this chart is replace the functions $x(t), x_T(t), X(j\omega)$, et cetera, with their "type signatures" (in the computer science sense). That is, instead of writing $x_T(t)$ for a CT periodic signal, we write $(S^1 \to \mathbb{C})$ to denote that it is a map from the circle to the complex numbers. Similarly, instead of writing $x[n]$ for a DT signal, we write $(\mathbb{Z} \to \mathbb{C})$ to denote that it is a map from the integers to the complex numbers.

Some things pop out when we write the Four Fourier transforms like this. First, both the CTFT and the DTFS preserve the type of function: the CTFT turns functions of a real variable into functions of a real variable, and the DTFS turns functions on the discrete circle to functions on the discrete circle. Second, although the CTFS and the DTFT do not have that property, they are intertwined in some way. The CT Fourier series turns functions on the circle into functions on the integers, while the DTFT does the opposite, turning functions on the integers into functions on the circle.

3.1 Inclusion and Approximation

Our ultimate goal is to relate these four transforms to one another. All we have done thus far is point out that they have some things in common. Let us start relating the types of functions to one another using *inclusion maps* and *approximation maps*.

Inclusion maps. Think back to how we showed that the CT Fourier series can be thought of as a special case of the CT Fourier transform. We noted that CT periodic signals are themselves regular-ole' CT signals, and can thus admit a notion of a Fourier transform. The Fourier transform of these signals turns out to be a sequence of impulses, which correspond exactly to the CT Fourier series.

Of course, this is a one-way street. It is not true that all CT signals are periodic, so we can't take the CT Fourier series of a general CT signal. To denote this, we will use something that we call an *inclusion map*. In the chart above, we denote signals/functions by their type signatures. To describe the idea that CT periodic signals are CT signals, but not the other way around, we will use a "hooked arrow," and write

$$(S^1 \to \mathbb{C}) \hookrightarrow (\mathbb{R} \to \mathbb{C}).$$

The "inclusion" transformation does nothing to the function that it is applied to: all it does is change its "type" to a less restrictive one.

Exercise 5: CTFS from CTFT

Consider some CT periodic signal $x_T(t)$. Derive its Fourier series in two different ways: one using the CT Fourier series directly, the other using the CT Fourier transform. Describe how to go from the CT Fourier series to the CT Fourier transform and back. (You don't actually need to do direct computations, just convince yourself they are the same.)

When you do the exercise above, another inclusion relationship will appear: one mapping the CT Fourier series coefficients to the CT Fourier transform. The notation for this one[2] is given by

$$(\mathbb{Z} \to \mathbb{C}) \hookrightarrow (\mathbb{R} \to \mathbb{C}).$$

Exercise 6: DT periodic signals as DT signals

Work out the inclusion map for DT periodic signals to treat them as DT signals. (Hint: the map looks like $(S^1_N \to \mathbb{C}) \hookrightarrow (\mathbb{Z} \to \mathbb{C})$.)

An important property of an inclusion map is a kind of invertibility. In general, there is no "inverse function" of an inclusion map: said maps are not necessarily *surjective*. However, inclusion maps are *injective*, meaning that distinct points in the domain are mapped to distinct points in the range.

Approximation maps. So periodic signals can be treated as regular signals by using inclusion maps, which gives us one potential way to dance around the rows of the chart we are building. There is another way that we can move around this chart: by taking limits.

Recall our initial derivation of the CT Fourier transform for finite duration signals. We did a "periodic extension" of that finite duration signal to build a periodic signal, and then took the Fourier series of that periodic signal. The Fourier transform was then defined as the limit of the Fourier series as the period is made larger and larger. We then argued that for signals with finite energy, this process doesn't break down too badly.

This procedure illustrates to us that general CT signals (at least, those with finite energy), can be approximated in some sense by periodic CT signals with extremely large period. Let us come up with another piece of notation to describe the idea of "taking a limit" in this way:

$$(\mathbb{R} \to \mathbb{C}) \rightsquigarrow (S^1 \to \mathbb{C}).$$

[2] Ok, so strictly speaking things with Dirac delta functions are not really functions of the type $(\mathbb{R} \to \mathbb{C})$. But just go with it, everything works out fine.

At first glance, it might look like we wrote this backwards: aren't we thinking of CT signals as limits of periodic CT signals? Shouldn't the arrow point the other way, then? In this case, no. All we want to describe is the idea that we can take a CT signal and approximate it *arbitrarily well* by a periodic CT signal. Doing that approximation takes a CT signal as input, and produces a periodic CT signal as output.

There are other reasonable approximation maps that you can build. Another one we will consider is in discrete-time:

$$(\mathbb{Z} \to \mathbb{C}) \rightsquigarrow (S^1_N \to \mathbb{C}).$$

In this case, N varies depending on how "good" you want your approximation to be.

Exercise 7: Approximation by sampling

We haven't learned about sampling yet, but it is reasonable to approximate a CT signal by a DT signal using evenly spaced samples.

1. Build up some conception of an approximation map for this procedure. Its type signature should look like $(\mathbb{R} \to \mathbb{C}) \rightsquigarrow (\mathbb{Z} \to \mathbb{C})$.
2. Do the same thing for sampling periodic CT signals: think of an approximation map that looks like $(S^1 \to \mathbb{C}) \rightsquigarrow (S^1_N \to \mathbb{C})$.
3. Does it make sense to compose these approximation maps to build new ones?
4. Think back to your intro course (ELEC241 at Rice) when the sampling theorem was introduced to you. Under what conditions can the sampling approximation maps be thought of as *exact*? That is, with no loss of information.

3.2 One Diagram to Rule Them All

We are now ready to draw one big picture that summarizes the Four Fourier transforms and the relationships between them. Figure 1 illustrates all of the relationships that we have built up. This is (something like) a *commutative diagram*. That is, you can follow the arrows on this diagram in a way such that the computation to get from one point to another does not depend on the path you take.

Fourier transforms take in functions of one type and put out functions of a (possibly different) type. By using inclusion and approximation maps, we can relate functions of different types to one another. Then, composing Fourier transforms with these maps lets us walk around this diagram however we please.

Recall our derivation of the CTFT from the CT Fourier series. The first step was to approximate a signal by a periodic one. Then we take the Fourier series. Then, we show that as the periodic approximation improves, the Fourier series converges in a way that resembles a Riemann sum. In some sense, Riemann sum approximations are the same as taking evenly spaced samples of a function. On the left column of the diagram, we approximated by doing a "periodic extension" of a signal with finite duration. On the right column, we approximated by taking evenly spaced samples of the Fourier representation. As an approximation on one side improves, so does the approximation on the other side: the Fourier transforms *couple* the approximation maps in this way.

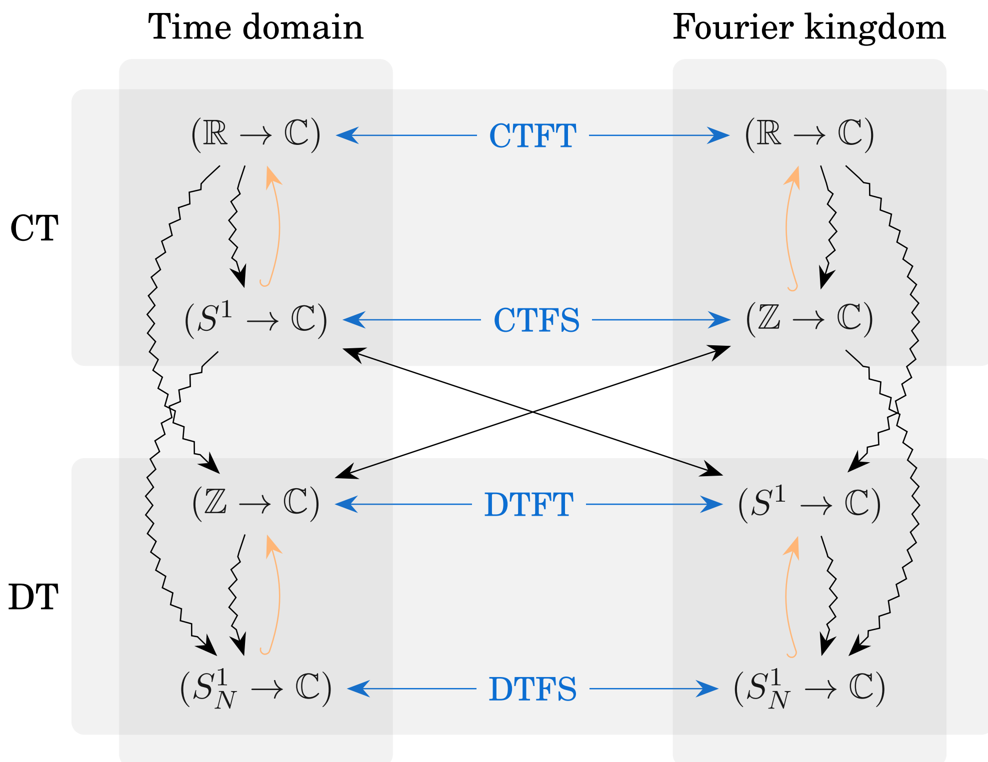

Figure 1: Chart of the Four Fourier transforms. Blue arrows denote Fourier transforms. Orange arrows denote inclusion maps. Black squiggly arrows denote approximation maps. Black straight arrows denote equality, namely that the two types of functions are the same.

Exercise 8: DTFT from the DTFS

Take the limit of the DTFS to get the DTFT, using the same method that we used to get the CTFT as the limit of the CTFS. Trace out how this works on the diagram.

I said that we can "walk around this diagram however we please," but there is one exception that comes about in the use of the diagonal arrows across the center.

Rule 1: Use of the Diagonal Arrows

From this diagram, the relationship between the CTFS and the DTFT is put on full display. Namely, they are basically the same thing! To see this, write down the analysis and synthesis equations for both and compare.

However, one needs to be careful with this. Indeed, this diagram allows you to jump around different types of functions by following the arrows, but the diagonal arrows need special care. The rule is this: if you cross between the time domain and the Fourier kingdom using one of the diagonal arrows, you need to do it a second time before you finish.

To see why this is the case, let us consider the relationship between periodic CT signals and the periodic functions given by the DTFT. Both have the "type signature" $(S^1 \to \mathbb{C})$, which allows us to move between the two with very little friction. However, the former is thought of as a "time function" and the latter is thought of as a "frequency function." When crossing from one to the other using the diagonal arrows, you can imagine that a citizen of the time domain is temporarily trespassing in the Fourier kingdom, and needs to return home at some point.

Examples. Let us now consider some examples. We will follow arrows on the diagram to show how some of the variants of the Fourier transforms can be derived in terms of the others.

Example 1: CTFT from the DTFS

The simplest Fourier transform, in some sense, is the DT Fourier series, and the most complicated is the CT Fourier transform. Let's build up the latter from the former.

Consider some CT signal $x(t)$. We can approximate it by a periodic DT signal by composing two approximation maps: call this new signal x_N, where N is determined by how good of an approximation we need. Then, we can compute the DTFS of x_N to get some a_N. Looking at the diagram, this is the same as computing the CTFT of $x(t)$ and then approximating it by some DT Fourier series $a_N[k]$.

Example 2: Duality

Remembering the distinction between the CT Fourier series and the DTFT is hard sometimes. Let us pick our favorite: since we learned the CT Fourier series first, we will go with that one. We will now use the diagram to give us instructions on computing the DTFT for some discrete-signal $x[n]$. First, we will "cross" from the time domain into the Fourier kingdom via the black diagonal arrow. This yields a Fourier series $a[k] := x[k]$. Then, we can use the synthesis equation for the CT Fourier series to construct a periodic CT signal $y(t)$ with period $T = 2\pi$. Then, we go back the Fourier kingdom via the black diagonal arrow, yielding the DTFT $X(j\omega) := y(\omega)$.

Exercise 9: Duality

Repeat the above example "Duality" to write the CT Fourier series in terms of the synthesis equation of the DTFT.

Example 3: "Inverting" the inclusion map

We pointed out that inclusion maps are injective, but not surjective. That is, there are only certain types of functions that you can define a unique inverse on. If we are careful, we might be able to leverage this to invert an inclusion map and prove nice properties of LTI systems.
Consider the LTI system described by convolution with a pulse train, that is, one with impulse response

$$h(t) = \sum_{k=-\infty}^{\infty} \delta(t-k).$$

We know that the CTFT of $h(t)$ is also a pulse train. Then, for some CT signal $x(t)$, we have, by the convolution theorem

$$(x \star h)(t) \xleftrightarrow{FT} 2\pi \sum_{k=-\infty}^{\infty} X(j2\pi k)\delta(\omega - 2\pi k).$$

Notice that the inclusion map $(\mathbb{Z} \to \mathbb{C}) \hookrightarrow (\mathbb{R} \to \mathbb{C})$ yields the same type of pulse train. In this case, we can turn $X(j\omega)H(j\omega)$ into a CT Fourier series $a[k]$, where $a[k] = 2\pi X(j2\pi k)$. That is, in this special case, we can *uniquely invert the inclusion map*. Applying the synthesis equation of the CT Fourier series yields a periodic signal that we will call $y(t)$ with period $T = 1$. Treating this periodic CT signal as a CT signal via the inclusion map $(S^1 \to \mathbb{C}) \hookrightarrow (\mathbb{R} \to \mathbb{C})$, we can confidently declare that the convolution $(x \star h)(t)$ yields a periodic signal.

Exercise 10: Convolution with a periodic signal

Repeat the above example of "Convolution with a pulse train" when $h(t)$ is a generic periodic signal, other than a pulse train. Namely, show that for $h(t)$ periodic, $(x \star h)(t)$ is periodic for any $x(t)$.

Exercise 11: Approximations of Inclusions

In the example "Convolution with a pulse train," we considered a special case in which an inclusion map can be inverted (due to being injective). For each of the inclusion maps in the diagram where an approximation map in the opposite direction is present, determine whether or not the approximation map is a suitable inverse for the inclusion map.

4 Index: The Sharp Bits

We index all of the maps pictured in Figure 1, apart from the Fourier transforms, which are covered in class.

4.1 Inclusion Maps

Using traditional function notation, we will use a lowercase iota to denote an inclusion map. For example $\iota : (S^1 \to \mathbb{C}) \hookrightarrow (\mathbb{R} \to \mathbb{C})$ describes treating a periodic CT signal as CT signal.

4.1.1 Periodic CT signals to CT signals

$$\begin{aligned} \iota : (S^1 \to \mathbb{C}) &\hookrightarrow (\mathbb{R} \to \mathbb{C}) \\ (\iota x_T)(t) &= x_T(t) \end{aligned}$$

4.1.2 CT Fourier series to CT Fourier transform

$$\begin{aligned} \iota : (\mathbb{Z} \to \mathbb{C}) &\hookrightarrow (\mathbb{R} \to \mathbb{C}) \\ (\iota a)(t) &= \sum_{k=-\infty}^{\infty} a[k]\delta(t - k2\pi/T) \end{aligned}$$

4.1.3 Periodic DT signals to DT signals

$$\begin{aligned} \iota : (S^1_N \to \mathbb{C}) &\hookrightarrow (\mathbb{Z} \to \mathbb{C}) \\ (\iota x_N)(n) &= x_N(t) \end{aligned}$$

4.1.4 DT Fourier series to DT Fourier transform

$$\begin{aligned} \iota : (S^1_N \to \mathbb{C}) &\hookrightarrow (S^1 \to \mathbb{C}) \\ (\iota a_N)(\omega) &= \sum_{k \in <N>} a_N[k]\delta(\omega - 2\pi k/N) \end{aligned}$$

4.2 Approximation Maps

We will use the letter Σ to denote a generic approximation map. For example, $\Sigma : (\mathbb{R} \to \mathbb{C}) \rightsquigarrow (\mathbb{Z} \to \mathbb{C})$ describes sampling a CT signal to get a DT signal.

4.2.1 CT signals to periodic CT signals

Pick some period $T > 0$, and approximate the signal $x(t)$ by a periodic signal as follows:

$$\begin{aligned} \Sigma : (\mathbb{R} \to \mathbb{C}) &\rightsquigarrow (S^1 \to \mathbb{C}) \\ (\Sigma x)(t) &= \begin{cases} x(t) & |t| \leq T/2 \\ \text{periodic} & \text{otherwise.} \end{cases} \end{aligned}$$

As $T \to \infty$, the approximation becomes better and better.

4.2.2 CT signals to DT signals

Pick some $T > 0$, and sample the signal $x(t)$ yielding

$$\begin{aligned} \Sigma &: (\mathbb{R} \to \mathbb{C}) \rightsquigarrow (\mathbb{Z} \to \mathbb{C}) \\ (\Sigma x)(n) &= x(nT). \end{aligned}$$

As $T \to 0$, the approximation becomes better and better.

4.2.3 Periodic CT signals to periodic DT signals

For a periodic CT signal $x_T(t)$, pick an integer $N > 0$, yielding

$$\begin{aligned} \Sigma &: (S^1 \to \mathbb{C}) \rightsquigarrow (S^1_N \to \mathbb{C}) \\ (\Sigma x_T)(n) &= x(nT/N). \end{aligned}$$

As $N \to \infty$, the approximation becomes better and better.

4.2.4 CT Fourier transform to CT Fourier series

Pick some $\Omega > 0$, and sample the function $X(j\omega)$ yielding

$$\begin{aligned} \Sigma &: (\mathbb{R} \to \mathbb{C}) \rightsquigarrow (\mathbb{Z} \to \mathbb{C}) \\ (\Sigma X)[k] &= X(jk\Omega). \end{aligned}$$

As $\Omega \to 0$, the approximation becomes better and better.

4.2.5 CT Fourier transform to DT Fourier transform

Pick some $\Omega > 0$, and approximate the function $X(j\omega)$ by a function on the circle as follows:

$$\begin{aligned} \Sigma &: (\mathbb{R} \to \mathbb{C}) \rightsquigarrow (S^1 \to \mathbb{C}) \\ (\Sigma X)(e^{j\omega}) &= \begin{cases} X(j\omega\Omega/(2\pi)) & -\pi \leq \omega < \pi \\ \text{periodic} & \text{otherwise.} \end{cases} \end{aligned}$$

As $\Omega \to \infty$, the approximation becomes better and better.

4.2.6 CT Fourier series to DT Fourier series

Pick some $N > 0$, and approximate the series $a[k]$ by a periodic DT function as follows:

$$\begin{aligned} \Sigma &: (\mathbb{Z} \to \mathbb{C}) \rightsquigarrow (S^1_N \to \mathbb{C}) \\ (\Sigma a)[k] &= \begin{cases} a[k] & |k| < N \\ \text{periodic} & \text{otherwise.} \end{cases} \end{aligned}$$

As $N \to \infty$, the approximation becomes better and better.

4.2.7 DT signals to periodic DT signals

Pick some $N > 0$, and approximate the DT signal $x[n]$ by a periodic DT signal as follows:

$$\begin{aligned}\Sigma : (\mathbb{Z} \to \mathbb{C}) &\rightsquigarrow (S^1_N \to \mathbb{C}) \\ (\Sigma x)[n] &= \begin{cases} x[n] & |n| < N \\ \text{periodic} & \text{otherwise.} \end{cases}\end{aligned}$$

As $N \to \infty$, the approximation becomes better and better.

4.2.8 DT Fourier transform to DT Fourier series

Pick an integer $N > 0$, and sample the function $X(e^{j\omega})$ yielding

$$\begin{aligned}\Sigma : (S^1 \to \mathbb{C}) &\rightsquigarrow (S^1_N \to \mathbb{C}) \\ (\Sigma X)[k] &= X(e^{jk2\pi/N}).\end{aligned}$$

As $N \to \infty$, the approximation becomes better and better.